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LEGENDRE FUNCTIONS WITH BOTH PARAMETERS LARGE

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CONTENTS

By application of the theory for second-order linear differential equations with two turning points developed in the preceding paper, some new asymptotic approximations are obtained for the associated Legendre functions when both the degree n and order m are large. The approximations are expressed in terms of parabolic cylinder functions, and are uniformly valid with respect to $x \in (-1, 1)$ and $m/(n+\frac{1}{2}) \in [\delta, 1+\Delta]$, where δ and Δ are arbitrary fixed numbers such that $0 < \delta < 1$ and $\Delta > 0$. The values of m and $n+\frac{1}{2}$ are either both real, or both purely imaginary. In all cases explicit bounds are supplied for the error terms associated with the approximations.

1. Introduction and summary

The purpose of this paper is to derive asymptotic approximations for solutions of the associated Legendre equation

$$(1-x^2)\frac{\mathrm{d}^2L}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}L}{\mathrm{d}x} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}L = 0 \tag{1.1}$$

when the parameters m and n are large, neither necessarily being an integer. For large n and fixed m (or more generally, bounded m) it is possible to construct asymptotic solutions in terms of Bessel functions of order m which are uniformly valid in unbounded x-domains containing one of the singularities ± 1 (see Olver 1974, chapter 12). In a similar way, for large m and fixed n the present writer has shown that it is possible to construct asymptotic solutions in terms of Bessel functions of order $n+\frac{1}{2}$ which are uniform in unbounded x-domains containing either $x = 1$ or $x = -1$. In this case, however, details have not been published.

When both m and n are large the problem is more difficult. The characterizing features of the differential equation in this situation are the regular singularities at $x = \pm 1$ and ∞, and the turning points at $x = \pm a$, where

$$1 - a^2 = m^2/(n+\tfrac{1}{2})^2. \tag{1.2}$$

Thorne (1957*b*) treated the case in which the ratio $m/(n+\frac{1}{2})$ has a given value in the open interval $(0, 1)$. He constructed asymptotic solutions in terms of Airy functions, and also in terms of Bessel functions of order m, these solutions being uniformly valid in a region that includes the interval $0 \leqslant x < 1$. When $m/(n+\frac{1}{2})$ is allowed to approach unity, however, Thorne's approximations no longer apply owing to lack of uniformity with respect to this ratio.

The present paper supplies asymptotic solutions of (1.1) for large values of $n+\frac{1}{2}$, complete with error bounds, which are uniformly valid when $m/(n+\frac{1}{2}) \in [\delta, 1+\Delta]$, where δ and Δ are any fixed numbers such that $0 < \delta < 1$ and $\Delta > 0$. Thus the turning points may be real and distinct $(0 < m < n+\frac{1}{2})$, coincident $(m = n+\frac{1}{2})$ or purely imaginary $(m > n+\frac{1}{2})$. The situation is exactly that covered by theorems I and II of the preceding paper (Olver 1975), and in §§2, 3 and 4 of the present paper standard solutions of (1.1) are identified in terms of the uniform asymptotic solutions involving parabolic cylinder functions. The results apply to the interval $-1 < x < 1$.

The remaining sections, §§5 and 6, are devoted to the case in which both m and $n+\frac{1}{2}$ are purely imaginary. Again there are either two real or two purely imaginary turning points, which coincide when $|m| = |n+\frac{1}{2}|$. Uniform asymptotic approximations, for the same interval $-1 < x < 1$, are constructed by application of theorems III and IV of the preceding paper.

All results presented are believed to be new.

2. Real parameters and real turning points: preliminary transformations

On removing the term in the first derivative from (1.1) by transformation of dependent variable, we obtain the differential equation

$$\frac{\mathrm{d}^2L}{\mathrm{d}x^2} = \left\{u^2\frac{x^2-a^2}{(1-x^2)^2} - \frac{3+x^2}{4(1-x^2)^2}\right\}L, \tag{2.1}$$

satisfied by $L = (1-x^2)^{\frac{1}{2}}P_n^m(x)$ and $(1-x^2)^{\frac{1}{2}}Q_n^m(x)$, in which

$$u = n+\tfrac{1}{2}, \quad a^2 = 1 - \frac{m^2}{u^2} = \frac{(n+m+\frac{1}{2})(n-m+\frac{1}{2})}{(n+\frac{1}{2})^2}; \tag{2.2}$$

compare (1.2). In the present section, and also §3, we suppose that

$$0 < m \leqslant n+\tfrac{1}{2},$$

so that $u > 0$ and $0 \leqslant a < 1$.

Equation (2.1) has singularities at $x = \pm 1$ and ∞, and turning points at $x = \pm a$. In each of the intervals $(1, \infty)$ and $(-\infty, -1)$, uniform asymptotic expansions of the solutions can be constructed in terms of elementary functions by standard methods.† We shall not pursue the details of those expansions; instead we confine attention to the interval $-1 < x < 1$.

Following Olver (1975, §2.2), we see that the appropriate Liouville transformation is given by

$$L = \left(\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)^{\frac{1}{2}} w, \quad \left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^2 = \frac{x^2-a^2}{(1-x^2)^2(\zeta^2-\alpha^2)}, \tag{2.3}$$

† Given, for example, in Olver (1974, chapters 6 and 10, especially § 5.3 of the former chapter).

where α is defined by

$$\int_{-\alpha}^{\alpha}(\alpha^2-\tau^2)^{\frac{1}{2}}\,d\tau = \int_{-a}^{a}\frac{(a^2-t^2)^{\frac{1}{2}}}{1-t^2}\,dt. \tag{2.4}$$

Evaluation of the last two integrals leads to the equation

$$\tfrac{1}{2}\alpha^2 = 1-(1-a^2)^{\frac{1}{2}}, \tag{2.5}$$

or, in terms of the original parameters,

$$\alpha^2 = 2(n-m+\tfrac{1}{2})/(n+\tfrac{1}{2}). \tag{2.6}$$

Thus α lies in the interval $[0,\sqrt{2})$, with $\alpha = 0$ corresponding to $a = 0$.

Since the points $\zeta = \pm\alpha$ correspond respectively to $x = \pm a$, integration of the second of (2.3) yields

$$\int_{\alpha}^{\zeta}(\tau^2-\alpha^2)^{\frac{1}{2}}\,d\tau = \int_{a}^{x}\frac{(t^2-a^2)^{\frac{1}{2}}}{1-t^2}\,dt \quad (a \leqslant x < 1), \tag{2.7}$$

$$\int_{\zeta}^{\alpha}(\alpha^2-\tau^2)^{\frac{1}{2}}\,d\tau = \int_{x}^{a}\frac{(a^2-t^2)^{\frac{1}{2}}}{1-t^2}\,dt \quad (-a \leqslant x \leqslant a), \tag{2.8}$$

and

$$\int_{\zeta}^{-\alpha}(\tau^2-\alpha^2)^{\frac{1}{2}}\,d\tau = \int_{x}^{-a}\frac{(t^2-a^2)^{\frac{1}{2}}}{1-t^2}\,dt \quad (-1 < x \leqslant -a). \tag{2.9}$$

From these equations and (2.4) it is evident that ζ is an even function of x, with $x = -1$, 0 and 1 corresponding to $\zeta = -\infty$, 0 and ∞, respectively. Moreover, (2.8) is replaceable by

$$\int_{0}^{\zeta}(\alpha^2-\tau^2)^{\frac{1}{2}}\,d\tau = \int_{0}^{x}\frac{(a^2-t^2)^{\frac{1}{2}}}{1-t^2}\,dt \quad (-a \leqslant x \leqslant a). \tag{2.10}$$

Evaluation of the integrals in (2.7) and (2.10) yields the formulae

$$\tfrac{1}{2}\zeta(\zeta^2-\alpha^2)^{\frac{1}{2}}-\tfrac{1}{2}\alpha^2\operatorname{arcosh}\left(\frac{\zeta}{\alpha}\right) = (1-a^2)^{\frac{1}{2}}\operatorname{artanh}\left\{\frac{1}{x}\left(\frac{x^2-a^2}{1-a^2}\right)^{\frac{1}{2}}\right\}-\operatorname{arcosh}\left(\frac{x}{a}\right) \quad (a \leqslant x < 1), \tag{2.11}$$

and

$$\tfrac{1}{2}\alpha^2\arcsin\left(\frac{\zeta}{\alpha}\right)+\tfrac{1}{2}\zeta(\alpha^2-\zeta^2)^{\frac{1}{2}} = \arcsin\left(\frac{x}{a}\right)-(1-a^2)^{\frac{1}{2}}\arctan\left\{x\left(\frac{1-a^2}{a^2-x^2}\right)^{\frac{1}{2}}\right\} \quad (-a \leqslant x \leqslant a), \tag{2.12}$$

in the case $a > 0$, or

$$\zeta^2 = -\ln(1-x^2) \quad (-1 < x < 1), \tag{2.13}$$

in the case $a = 0$.

The transformed differential equation is given by

$$d^2w/d\zeta^2 = \{u^2(\zeta^2-\alpha^2)+\psi(\alpha,\zeta)\}\,w, \tag{2.14}$$

where

$$\psi(\alpha,\zeta) = -\dot{x}^2\frac{3+x^2}{4(1-x^2)^2}+\dot{x}^{\frac{1}{2}}\frac{d^2}{d\zeta^2}(\dot{x}^{-\frac{1}{2}}),$$

dots signifying differentiations with respect to ζ; compare Olver (1975, (2.10)). Using the second of (2.3) and carrying out the differentiations, we arrive at the explicit formula

$$\psi(\alpha,\zeta) = \frac{3\zeta^2+2\alpha^2}{4(\zeta^2-\alpha^2)^2}+\frac{\zeta^2-\alpha^2}{4(x^2-a^2)^3}\{(3-4a^2)\,x^4-(3-6a^2+a^4)\,x^2-a^2(2-a^2)\}. \tag{2.15}$$

Lemma I of Olver (1975, §3) shows that $\psi(\alpha,\zeta)$ is continuous in the region $\alpha\in[0,\sqrt{2})$, $\zeta\in(-\infty,\infty)$.

In identifying the solutions of (2.14) we shall need to know the behaviour of $\psi(\alpha,\zeta)$ as ζ tends to its extreme values. When $\zeta\to\infty$ and $x\to 1-$, we derive from (2.11)

$$\zeta^2-\tfrac{1}{2}\alpha^2-\alpha^2\ln\left(\frac{2\zeta}{\alpha}\right)+O\left(\frac{1}{\zeta^2}\right) = (1-a^2)^{\frac{1}{2}}\ln\left\{\frac{2(1-a^2)}{a^2(1-x)}\right\}-2\ln\left\{\frac{1+(1-a^2)^{\frac{1}{2}}}{a}\right\}+O(1-x).$$

By reversion and use of (2.5) we find that

$$(1-\tfrac{1}{2}\alpha^2)\ln\left(\frac{1}{1-x}\right) = \zeta^2-\alpha^2\ln\zeta+\ln\hat{\alpha}+O\left(\frac{1}{\zeta^2}\right) \quad (\zeta\to\infty), \tag{2.16}$$

where

$$\hat{\alpha} = \left(\frac{a}{\alpha}\right)^{4-\alpha^2}\frac{2^{\frac{1}{2}(6-3\alpha^2)}}{(2-\alpha^2)^{2-\alpha^2}}\mathrm{e}^{-\frac{1}{2}\alpha^2}. \tag{2.17}$$

Exponentiation of (2.16) yields

$$x = 1-\left\{\exp\left(-\frac{2\zeta^2}{2-\alpha^2}\right)\right\}\left(\frac{\zeta^{\alpha^2}}{\hat{\alpha}}\right)^{2/(2-\alpha^2)}\left\{1+O\left(\frac{1}{\zeta^2}\right)\right\}. \tag{2.18}$$

When $x = 1$ the content of the braces in (2.15) vanishes identically. Therefore as $\zeta\to+\infty$

$$\psi(\alpha,\zeta) = \frac{3\zeta^2+2\alpha^2}{4(\zeta^2-\alpha^2)^2}+O\left\{\zeta^{4/(2-\alpha^2)}\exp\left(-\frac{2\zeta^2}{2-\alpha^2}\right)\right\}.$$

Since $\psi(\alpha,\zeta)$ is even in ζ, this result also holds when $\zeta\to-\infty$. Thus we have

$$\psi(\alpha,\zeta) = \frac{3}{4\zeta^2}+O\left(\frac{1}{\zeta^4}\right) \quad (\zeta\to\pm\infty). \tag{2.19}$$

Moreover, from the foregoing analysis it is clear that this result is uniformly valid for $\alpha\in[0,\mathrm{A}]$, where—in this section and the next—A denotes an arbitrary constant in the interval $(0,\sqrt{2})$.

It should be observed that the fact that the order of $\psi(\alpha,\zeta)$ is as small as $O(\zeta^{-2})$ as $\zeta\to\pm\infty$ is not entirely fortuitous. It is a consequence of the choice of the large parameter made at the beginning of this section. Had we made, for example, the quite natural choice $u = \{n(n+1)\}^{\frac{1}{2}}$ in place of the first of (2.2), then in the expression corresponding to (2.15) the content of the braces would not vanish as $x\to 1$, causing $\psi(\alpha,\zeta)$ to behave as a multiple of ζ^2 as $\zeta\to\pm\infty$. In turn, this would have the undesirable consequence of restricting the validity of the resulting asymptotic solutions of the differential equation to bounded values of ζ (compare also Olver 1974, pp. 463–4 and Olver 1975, § 6.3).

3. Real parameters and real turning points: identification of solutions

Let A denote any constant in the interval $(0,\sqrt{2})$. On applying theorem I of Olver (1975, § 6), with $\zeta_2=\infty$, we obtain the following solutions of equation (2.14):

$$w_1(u,\alpha,\zeta) = U(-\tfrac{1}{2}u\alpha^2,\zeta\sqrt{2u})+\epsilon_1(u,\alpha,\zeta), \quad w_2(u,\alpha,\zeta) = \bar{U}(-\tfrac{1}{2}u\alpha^2,\zeta\sqrt{2u})+\epsilon_2(u,\alpha,\zeta), \tag{3.1}$$

valid when $\zeta\in[0,\infty)$ and $\alpha\in[0,\mathrm{A}]$, where U and $\bar{U}$ denote the parabolic cylinder functions defined and discussed in Miller (1955) and Olver (1975). The solutions $w_1(u,\alpha,\zeta)$ and $w_2(u,\alpha,\zeta)$ now need to be identified in terms of standard solutions of (2.1).

As $\zeta\to\infty$, $w_1(u,\alpha,\zeta)$ is recessive and $w_2(u,\alpha,\zeta)$ is dominant. Hence $w_1(u,\alpha,\zeta)$ is a multiple of

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the solution of (2.1) that is recessive as $x \to 1-$. This solution is $(1-x^2)^{\frac{1}{2}} \mathrm{P}_n^{-m}(x)$, where $\mathrm{P}_n^{-m}(x)$ denotes the Ferrers function of degree n and order $-m$.† Thus from (2.3)

$$w_1(u, \alpha, \zeta) = \{(x^2-a^2)/(\zeta^2-\alpha^2)\}^{\frac{1}{4}} S \mathrm{P}_n^{-m}(x), \tag{3.2}$$

where the coefficient S is independent of x. The value of S can be found by evaluating the ratio of the two sides of this equation as $x \to 1-$. First, we have

$$\mathrm{P}_n^{-m}(x) \sim \frac{1}{\Gamma(m+1)} \left(\frac{1-x}{2}\right)^{\frac{1}{2}m}.$$

Next, as $\zeta \to \infty$

$$U(-\tfrac{1}{2}u\alpha^2, \zeta\sqrt{2u}) \sim (\zeta\sqrt{2u})^{\frac{1}{2}(u\alpha^2-1)} \mathrm{e}^{-\frac{1}{2}u\zeta^2},$$

and from the bound (6.5) of Olver (1975) for $\epsilon_1(u, \alpha, \zeta)$ it is seen that the ratio of this error term to $U(-\frac{1}{2}u\alpha^2, \zeta\sqrt{2u})$ vanishes. Therefore

$$S = \lim_{\zeta\to\infty} \left\{\Gamma(m+1) \left(\frac{2}{1-x}\right)^{\frac{1}{2}m} \frac{\zeta^{\frac{1}{2}u\alpha^2}(2u)^{\frac{1}{4}(u\alpha^2-1)}}{(1-a^2)^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2}u\zeta^2}}\right\}.$$

On using (2.2), (2.6), (2.16) and (2.17), we arrive at

$$S = \left(\frac{n+\frac{1}{2}}{2}\right)^{\frac{1}{4}} \frac{\Gamma(m+1)}{m^{m+\frac{1}{2}}} \frac{(n+m+\frac{1}{2})^{\frac{1}{2}(2n+2m+1)}}{\mathrm{e}^{\frac{1}{2}(2n-2m+1)}}. \tag{3.3}$$

Combination of (3.2), (3.3) and the first of (3.1) yields the first of the desired approximations:

$$\mathrm{P}_n^{-m}(x) = \left(\frac{2}{n+\frac{1}{2}}\right)^{\frac{1}{4}} \frac{m^{m+\frac{1}{2}}}{\Gamma(m+1)} \frac{\mathrm{e}^{\frac{1}{2}(2n-2m+1)}}{(n+m+\frac{1}{2})^{\frac{1}{2}(2n+2m+1)}} \left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}} \{U(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})+\epsilon_1\}, \tag{3.4}$$

valid when $0 < m \leqslant n+\frac{1}{2}$ and $0 \leqslant x < 1$. In this result, the variable ζ is given by (2.11) and (2.12), and the error term ϵ_1 bounded by (6.5) of Olver (1975).

To show that $|\epsilon_1|$ is small compared with $|U(m-n-\frac{1}{2}, \zeta\sqrt{2n+1})|$ when n is large, we now obtain an asymptotic estimate for ϵ_1 which is uniform with respect to x and the ratio $m/(n+\frac{1}{2})$ provided that

$$\delta(n+\tfrac{1}{2}) \leqslant m \leqslant n+\tfrac{1}{2}, \tag{3.5}$$

where δ is any fixed number in the interval (0, 1).‡ Then $0 \leqslant \alpha \leqslant A$, where A is a fixed number in $(0, \sqrt{2})$. Applying the results of § 6.3 of Olver (1975) with $\Omega(x)$ set equal to $|x|^{\frac{1}{3}}$—a substitution that is permissible as a consequence of (2.19)—we derive

$$\begin{aligned}\epsilon_1 &= \mathsf{E}^{-1}(-\tfrac{1}{2}u\alpha^2, \zeta\sqrt{2u})\, \mathsf{M}(-\tfrac{1}{2}u\alpha^2, \zeta\sqrt{2u})\, O(u^{-\frac{2}{3}}) \\ &= \mathsf{E}^{-1}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\, \mathsf{M}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\, O(n^{-\frac{2}{3}}),\end{aligned}$$

where E and M are the auxiliary functions introduced in § 5.8 of Olver (1975). Next, on using Stirling's formula we obtain from (3.3)

$$S = (n+\tfrac{1}{2})^{\frac{1}{4}} 2^{\frac{1}{2}(n+m)} \Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{3}{4}) \{1+O(n^{-1})\}.$$

† For definitions and properties of Ferrers functions, or 'Legendre functions on the cut', the reader is referred to B.M.P. (1953, §3.4), or Olver (1974, pp. 185–9).

‡ The point of the restriction $\delta(n+\frac{1}{2}) \leqslant m$ is to ensure that the turning point at $x = a$ is bounded away from the singularity at $x = 1$. A coalescing turning point and pole cannot be included in the present theory, but could be treated by an extension of the theory of Thorne (1957*a*, *b*).

Combination of these results with (3.1) and (3.2) yields

$$\mathrm{P}_n^{-m}(x) = \frac{1}{(n+\frac{1}{2})^{\frac{1}{4}}\,2^{\frac{1}{2}(n+m)}\,\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{3}{4})}\left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}}$$
$$\times\{U(m-n-\tfrac{1}{2},\zeta\sqrt{2n+1})+\mathsf{E}^{-1}(m-n-\tfrac{1}{2},\zeta\sqrt{2n+1})\,\mathsf{M}(m-n-\tfrac{1}{2},\zeta\sqrt{2n+1})\,O(n^{-\frac{2}{3}})\} \quad (3.6)$$

as $n\to\infty$, uniformly with respect to $x\in[0,1)$ and values of m satisfying (3.5). Except in the neighbourhoods of zeros of $U(m-n-\frac{1}{2},\zeta\sqrt{2n+1})$, the relative error of this asymptotic approximation to $\mathrm{P}_n^{-m}(x)$ is uniformly $O(n^{-\frac{2}{3}})$.

Equations (3.4) and (3.6) have been derived by identification of solutions at the singularity $x=1$. At the other endpoint of the interval of validity it is known that

$$\mathrm{P}_n^{-m}(0) = \frac{\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})\cos(\frac{1}{2}n\pi-\frac{1}{2}m\pi)}{\pi^{\frac{1}{2}}2^m\Gamma(\frac{1}{2}n+\frac{1}{2}m+1)}, \quad (3.7)$$

and it is of obvious interest to compare this result with the form of (3.6) at $x=\zeta=0$. We have

$$\left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}} = \left(\frac{\alpha}{a}\right)^{\frac{1}{2}} = \left(\frac{2n+1}{n+m+\frac{1}{2}}\right)^{\frac{1}{4}}.$$

Also, $$U(m-n-\tfrac{1}{2},0) = \pi^{-\frac{1}{2}}2^{\frac{1}{2}(n-m)}\,\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})\cos(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi),$$

and $$\mathsf{E}^{-1}(m-n-\tfrac{1}{2},0)\,\mathsf{M}(m-n-\tfrac{1}{2},0) = \pi^{-\frac{1}{2}}2^{\frac{1}{2}(n-m)}\,\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2}).$$

Accordingly, the right-hand side of (3.6) reduces to

$$\frac{\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})}{\pi^{\frac{1}{2}}2^m(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{4})^{\frac{1}{4}}\,\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{3}{4})}\{\cos(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi)+O(n^{-\frac{2}{3}})\}. \quad (3.8)$$

By means of Stirling's formula it is easily verified that (3.7) agrees with (3.8) within the tolerance of the uniform error term, thus providing a powerful check on the soundness of our asymptotic theory.

To identify the second solution, write

$$w_2(u,\alpha,\zeta) = \{(x^2-a^2)/(\zeta^2-\alpha^2)\}^{\frac{1}{4}}\{S_1\mathrm{P}_n^{-m}(x)+S_2\mathrm{Q}_n^{-m}(x)\}, \quad (3.9)$$

where $\mathrm{Q}_n^{-m}(x)$ denotes the second Ferrers function, and S_1 and S_2 are independent of x. A convenient way of determining S_1 and S_2 is to compare the two sides of (3.9), together with their differentiated forms, at $x=\zeta=0$. On the right-hand side, we have (3.7) and the corresponding expressions

$$\mathrm{P}_n^{-m\prime}(0) = \frac{\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\sin(\frac{1}{2}n\pi-\frac{1}{2}m\pi)}{\pi^{\frac{1}{2}}2^{m-1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})},$$

$$\mathrm{Q}_n^{-m}(0) = -\frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})\sin(\frac{1}{2}n\pi-\frac{1}{2}m\pi)}{2^{m+1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+1)}, \quad \mathrm{Q}_n^{-m\prime}(0) = \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\cos(\frac{1}{2}n\pi-\frac{1}{2}m\pi)}{2^m\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})}.$$

On the left-hand side,

$$\bar{U}(-\tfrac{1}{2}u\alpha^2,0) = -\pi^{-\frac{1}{2}}2^{\frac{1}{2}(n-m)}\,\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})\sin(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi),$$
$$\bar{U}'(-\tfrac{1}{2}u\alpha^2,0) = \pi^{-\frac{1}{2}}2^{\frac{1}{2}(n-m+1)}\,\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+1)\cos(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi),$$

and $\epsilon_2=\partial\epsilon_2/\partial\zeta=0$ at $\zeta=0$. Solution of the two linear algebraic equations for S_1 and S_2 yields the required result:

$$S_1 = 2^{\frac{1}{2}(n+m-2)}(n+\tfrac{1}{2})^{\frac{1}{4}}\sin(n\pi-m\pi)$$
$$\times\{(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{4})^{\frac{1}{4}}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2})-(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{4})^{-\frac{1}{4}}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+1)\}, \quad (3.10)$$

$$S_2 = \pi^{-1}\, 2^{\frac{1}{2}(n+m+2)}(n+\tfrac{1}{2})^{\frac{1}{4}}\{(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{4})^{\frac{1}{4}}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2})\cos^2(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi) + (\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{4})^{-\frac{1}{4}}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+1)\sin^2(\tfrac{1}{2}n\pi-\tfrac{1}{2}m\pi)\}. \quad (3.11)$$

For large n, we find that

$$S_2 = \pi^{-1}\, 2^{\frac{1}{2}(n+m+2)}(n+\tfrac{1}{2})^{\frac{1}{4}}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{3}{4})\{1+O(n^{-1})\}, \quad S_1/S_2 = O(n^{-1}),$$

the O-terms being uniform for values of m satisfying (3.5). Combining these estimates with (3.6) and referring to properties of the auxiliary functions

$$\mathsf{E}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1}) \quad \text{and} \quad \mathsf{M}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})$$

given in Olver (1975, § 5.8) (including the fact that the former equals or exceeds unity), we see that the term

$$\{(x^2-a^2)/(\zeta^2-\alpha^2)\}^{\frac{1}{4}}\, S_1\, \mathrm{P}_n^{-m}(x)$$

on the right-hand side of (3.9) is absorbable in the estimate

$$\mathsf{E}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\,\mathsf{M}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\,O(n^{-\frac{2}{3}})$$

for the error term $\epsilon_2(u, \alpha, \zeta)$ associated with the left-hand side. Thus we arrive at the following uniform asymptotic approximation for large n, valid with the same conditions as (3.6):

$$\mathrm{Q}_n^{-m}(x) = \frac{\pi}{(n+\frac{1}{2})^{\frac{1}{4}}\, 2^{\frac{1}{2}(n+m+2)}\,\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{3}{4})}\left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}} \times\{\bar{U}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1}) + \mathsf{E}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\,\mathsf{M}(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})\,O(n^{-\frac{2}{3}})\}. \quad (3.12)$$

A check on (3.12) is to compare the limiting forms of the two sides of this equation as $x \to 1-$, that is, as $\zeta \to \infty$. Again, it is found that there is agreement within the tolerance of the given error term.

Relations (3.6) and (3.12) apply when $x \in [0, 1)$. Corresponding results for the interval $(-1, 0]$ are derivable from § 6.4 of Olver (1975), or by use of the well-known connection formulae for the Ferrers functions. Connection formulae can also be used to derive corresponding approximations for other Ferrers functions, including $\mathrm{P}_n^m(x)$ and $\mathrm{Q}_n^m(x)$. Details are straightforward and need not be recorded.

Some previous asymptotic approximations for associated Legendre functions are included as special cases of the present results. For example, if $m/(n+\frac{1}{2})$ has a *fixed* value in the interval (0, 1), then $m-n-\frac{1}{2}$ tends to $-\infty$ as $n+\frac{1}{2} \to \infty$. In this case the parabolic cylinder functions in (3.6) and (3.12) may be replaced by their asymptotic approximations in terms of Airy functions given, for example, in Olver (1975, § 5.3). The results then reduce to Thorne's (1957*b*) asymptotic approximations for Ferrers functions in terms of Airy functions, truncated at their first term.† The unavoidable price of the extensive uniformity of the present results is the complexity of the relations (2.11) and (2.12) between the variables ζ and x.

† Compare also the non-uniform asymptotic approximations given in § 5 of Brussaard & Tolhoek (1957) for the similar problem of the eigenfunctions of a symmetric rotator.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

4. Real parameters and imaginary turning points

In this section we consider equation (1.1) with the conditions

$$0 < n+\tfrac{1}{2} \leqslant m.$$

Corresponding to (2.2), we set

$$u = n+\tfrac{1}{2}, \quad a^2 = \frac{m^2}{u^2}-1 = \frac{(n+m+\frac{1}{2})(m-n-\frac{1}{2})}{(n+\frac{1}{2})^2}, \tag{4.1}$$

so that $u > 0$ and $a \geqslant 0$. Then we have

$$\frac{d^2L}{dx^2} = \left\{u^2\frac{x^2+a^2}{(1-x^2)^2}-\frac{3+x^2}{4(1-x^2)^2}\right\}L, \tag{4.2}$$

with solutions $L = (1-x^2)^{\frac{1}{2}}\mathrm{P}_n^m(x)$ and $(1-x^2)^{\frac{1}{2}}\mathrm{Q}_n^m(x)$.

The turning points of the differential equation (4.2) are located at $x = \pm ia$. The appropriate Liouville transformation is therefore that of § 2.3 of Olver (1975). The results of § 2 above again apply, provided that a^2 and α^2 are replaced throughout by $-a^2$ and $-\alpha^2$, respectively. Thus

$$\tfrac{1}{2}\alpha^2 = (1+a^2)^{\frac{1}{2}}-1 = (m-n-\tfrac{1}{2})/(n+\tfrac{1}{2}), \tag{4.3}$$

and

$$\int_0^{\zeta}(\alpha^2+\tau^2)^{\frac{1}{2}}\,d\tau = \int_0^x\frac{(a^2+t^2)^{\frac{1}{2}}}{1-t^2}\,dt,$$

that is,

$$\tfrac{1}{2}\zeta(\zeta^2+\alpha^2)^{\frac{1}{2}}+\tfrac{1}{2}\alpha^2\,\mathrm{arsinh}\left(\frac{\zeta}{\alpha}\right) = (1+a^2)^{\frac{1}{2}}\,\mathrm{artanh}\left\{x\left(\frac{1+a^2}{x^2+a^2}\right)^{\frac{1}{2}}\right\}-\mathrm{arsinh}\left(\frac{x}{a}\right). \tag{4.4}$$

The last two equations hold when $x\in(-1, 1)$, or, equivalently, $\zeta\in(-\infty, \infty)$. When $a = 0$, (4.4) is replaced by (2.13).

With $L = (1-x^2)^{\frac{1}{2}}(\zeta^2+\alpha^2)^{\frac{1}{4}}(x^2+a^2)^{-\frac{1}{4}}w$, the transformed differential equation is given by

$$d^2w/d\zeta^2 = \{u^2(\zeta^2+\alpha^2)+\psi(\alpha,\zeta)\}\,w, \tag{4.5}$$

in which

$$\psi(\alpha,\zeta) = \frac{3\zeta^2-2\alpha^2}{4(\zeta^2+\alpha^2)^2}+\frac{\zeta^2+\alpha^2}{4(x^2+a^2)^3}\{(3+4a^2)\,x^4-(3+6a^2+a^4)\,x^2+a^2(2+a^2)\}; \tag{4.6}$$

compare (2.15). Lemma II of Olver (1975, § 4.1) shows that $\psi(\alpha,\zeta)$ is continuous in the region $\alpha\in[0,\infty)$, $\zeta\in(-\infty,\infty)$. And in the manner of § 2 above it is verifiable that

$$\psi(\alpha,\zeta) = \frac{3}{4\zeta^2}+O\left(\frac{1}{\zeta^4}\right) \tag{4.7}$$

as $\zeta\rightarrow\pm\infty$, uniformly for $\alpha\in[0, \mathrm{A}]$, where A now denotes *any* fixed positive number.

For large u, approximate solutions of (4.5) are supplied by theorem II of Olver (1975, § 7.1). They are identifiable in terms of the Ferrers functions in the manner of § 3. Corresponding to (3.4), we have

$$\mathrm{P}_n^{-m}(x) = \left(\frac{2}{n+\frac{1}{2}}\right)^{\frac{1}{4}}\frac{m^{m+\frac{1}{2}}}{\Gamma(m+1)}\,\frac{e^{\frac{1}{4}(2n-2m+1)}}{(n+m+\frac{1}{2})^{\frac{1}{4}(2n+2m+1)}}\left(\frac{\zeta^2+\alpha^2}{x^2+a^2}\right)^{\frac{1}{4}}\{U(m-n-\tfrac{1}{2}, \zeta\sqrt{2n+1})+\epsilon\}, \tag{4.8}$$

where the error term is subject to the bound (7.3) of Olver (1975), with $\zeta_2 = \infty$. This bound is valid when $0 < n+\frac{1}{2} \leqslant m$ and $-1 < x < 1$.

The asymptotic result corresponding to (3.6) is available from §7.3 of the last reference. Using Stirling's formula we obtain

$$\mathrm{P}_n^{-m}(x) = \frac{1}{(n+\frac{1}{2})^{\frac{1}{4}}\,2^{\frac{1}{2}(n+m)}\,\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{3}{4})}\left(\frac{\zeta^2+\alpha^2}{x^2+a^2}\right)^{\frac{1}{4}} U(m-n-\tfrac{1}{2},\,\zeta\sqrt{2n+1})\,\{1+O(n^{-1}\ln n)\} \qquad (4.9)$$

as $n\to\infty$, uniformly for $-1 < x < 1$ and

$$n+\tfrac{1}{2} \leqslant m \leqslant (1+\Delta)(n+\tfrac{1}{2}),$$

Δ denoting any positive constant.

A check on (4.9) is to let $\zeta\to-\infty$ and compare the limiting form of the right-hand side with the relation

$$\mathrm{P}_n^{-m}(x) \sim \frac{\Gamma(m)}{\Gamma(m-n)\,\Gamma(n+m+1)}\left(\frac{2}{1+x}\right)^{\frac{1}{2}m} \qquad (x\to-1+).$$

Agreement is obtained within the tolerance of the error term in (4.9).

Since the relations (4.8) and (4.9) apply to negative values of x as well as to positive values, they furnish uniform asymptotic approximations for the solutions $\mathrm{P}_n^{-m}(x)$ and $\mathrm{P}_n^{-m}(-x)$ of equation (1.1). With the given conditions these two solutions are independent and numerically satisfactory; therefore similar uniform asymptotic approximations of any other solutions can be constructed by use of connection formulae. In particular, this includes $\mathrm{P}_n^{m}(x)$, $\mathrm{Q}_n^{-m}(x)$ and $\mathrm{Q}_n^{m}(x)$.

Comments similar to those made in the final paragraph of §3 apply also to the results obtained in the present section.

5. Imaginary parameters and real turning points

This case is similar to that of §§2 and 3, the essential changes being that u and m are replaced throughout by $\mathrm{i}u$ and $\mathrm{i}m$, respectively. Thus we are now considering the Ferrers function $\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(x)$ with $0 < m \leqslant u$ and $-1 < x < 1$.

Again we write $a^2 = 1-(m^2/u^2)$, and except for (2.6) the transformations (2.3) to (2.13) apply unchanged. Corresponding to (2.14) we have the differential equation

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 = \{-u^2(\zeta^2-\alpha^2)+\psi(\alpha,\zeta)\}\,w, \qquad (5.1)$$

with solutions

$$w = \{(x^2-a^2)/(\zeta^2-\alpha^2)\}^{\frac{1}{4}}\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(\pm x),$$

the function $\psi(\alpha,\zeta)$ being given by (2.15).

For large u, approximate solutions of (5.1), in terms of modified parabolic cylinder functions, are supplied by theorem III of Olver (1975, §9.1), with $\zeta_2 = \infty$. As in §3 above, the first of these solutions is easily related to one of the standard solutions by considering limiting behaviour as $x\to1-$, that is, as $\zeta\to\infty$. The second approximate solution is identifiable by using the known values of the Ferrers functions and their derivatives at $x = 0$; in this case the analysis is facilitated by the identities given in Miller (1955, §9.3) for Gamma functions of arguments $\frac{1}{4}s+\frac{1}{2}\mathrm{i}a$, $s = 0, 1, 2, 3, 4$. The final results are found to be expressible in the form

$$\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(x)\} = \frac{m^{\frac{1}{2}}}{(2u)^{\frac{1}{4}}}\left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}}\{k^{-\frac{1}{2}}(u-m)\,W(u-m,\,\zeta\sqrt{2u})+\epsilon_1\}, \qquad (5.2)$$

and

$$\left(\frac{1}{\mathrm{Re}\,T_2}+\frac{1}{\mathrm{Re}\,T_3}\right)\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(x)\}+\left(\frac{1}{\mathrm{Re}\,T_2}-\frac{1}{\mathrm{Re}\,T_3}\right)\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(-x)\}$$
$$= \left(\frac{\zeta^2-\alpha^2}{x^2-a^2}\right)^{\frac{1}{4}}\{k^{\frac{1}{2}}(u-m)\,W(u-m,\,-\zeta\sqrt{2u})+\epsilon_2\}, \qquad (5.3)$$

where W denotes the modified parabolic cylinder function defined and discussed in Miller (1955) and Olver (1975), and T_1, T_2 and T_3 are defined by

$$T_1 = T\exp\{\tfrac{1}{2}\mathrm{i}\,\mathrm{ph}\,\Gamma(\tfrac{1}{2}+\mathrm{i}u-\mathrm{i}m)\},$$

$$T_2 = 2\left\{\frac{2(u+m)\cosh(\pi u-\pi m)}{uk^2(u-m)}\right\}^{\frac{1}{4}}\frac{2^{\frac{1}{2}(u+m)\mathrm{i}}T}{\Gamma(\frac{3}{4}-\frac{1}{2}\mathrm{i}u-\frac{1}{2}\mathrm{i}m)}\exp\{-\tfrac{1}{2}\mathrm{i}\arctan\tanh(\tfrac{1}{2}\pi u-\tfrac{1}{2}\pi m)\},$$

$$T_3 = -4\left\{\frac{\cosh(\pi u-\pi m)}{2u(u+m)\,k^2(u-m)}\right\}^{\frac{1}{4}}\frac{2^{\frac{1}{2}(u+m)\mathrm{i}}T}{\Gamma(\frac{1}{4}-\frac{1}{2}\mathrm{i}u-\frac{1}{2}\mathrm{i}m)}\exp\{\tfrac{1}{2}\mathrm{i}\arctan\tanh(\tfrac{1}{2}\pi u-\tfrac{1}{2}\pi m)\},$$

with
$$T \equiv m^{\mathrm{i}m}\Gamma(1-\mathrm{i}m)\,(u+m)^{-\frac{1}{2}(u+m)\mathrm{i}}\,\mathrm{e}^{\frac{1}{4}(2u-2m+\pi)\mathrm{i}}. \tag{5.4}$$

All functions in the foregoing equations take their principal values, except $\mathrm{ph}\,\Gamma(\frac{1}{2}+\mathrm{i}u-\mathrm{i}m)$, which is zero when $u-m$ is zero and defined by continuity for other values of $u-m$. The error terms ϵ_1 and ϵ_2 are bounded as in (9.4) and (9.5) of Olver (1975), these bounds applying when $0 < m \leqslant u$ and $0 \leqslant x < 1$.

For large u, asymptotic estimates of ϵ_1 in the ζ-intervals $[0, \infty)$ and $(-\infty, 0]$ are immediately available from the results of §§ 9.3 and 9.4 of Olver (1975). Thus

$$\epsilon_1 = \mathsf{E}^{-1}(u-m, \zeta\sqrt{2u})\,\mathsf{M}(u-m, \zeta\sqrt{2u})\,O(u^{-\frac{2}{3}}) \quad (0 \leqslant \zeta < \infty), \tag{5.5}$$

and
$$\epsilon_1 = k^{-1}(u-m)\,\mathsf{E}(u-m, \zeta\sqrt{2u})\,\mathsf{M}(u-m, \zeta\sqrt{2u})\,O(u^{-\frac{2}{3}}) \quad (-\infty < \zeta \leqslant 0). \tag{5.6}$$

In each case the O-term is uniform with respect to ζ in the given interval and $m \in [\delta u, u]$, δ again denoting a fixed number in the interval $(0, 1)$. The auxiliary functions E and M are defined as in § 8.6 of the same reference.

It is of interest to notice that the result (5.6) is also derivable from (5.3), as follows. Substituting in (5.4) by means of Stirling's formula, we obtain

$$T = (2\pi m)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}m\pi}\,\mathrm{e}^{\frac{1}{2}(u+m)\mathrm{i}}\,(u+m)^{-\frac{1}{2}(u+m)\mathrm{i}}\{1+O(m^{-1})\} \quad (m\to\infty).$$

Then using this result and the definition

$$k(u-m) = \{1+\mathrm{e}^{2\pi(u-m)}\}^{\frac{1}{2}} - \mathrm{e}^{\pi(u-m)},$$

we find that for large u and $m \in [\delta u, u]$ the quantities $\mathrm{Re}\,T_2$ and $\mathrm{Re}\,T_3$ are approximated uniformly by

$$\mathrm{Re}\,T_2 = \frac{2m^{\frac{1}{2}}}{(2u)^{\frac{1}{4}}k(u-m)}\left\{1+O\left(\frac{1}{u}\right)\right\}, \quad \mathrm{Re}\,T_3 = -\frac{2m^{\frac{1}{2}}}{(2u)^{\frac{1}{4}}k(u-m)}\left\{1+O\left(\frac{1}{u}\right)\right\}.$$

Accordingly,
$$\frac{1}{\mathrm{Re}\,T_2}-\frac{1}{\mathrm{Re}\,T_3} = (2u)^{\frac{1}{4}}m^{-\frac{1}{2}}k(u-m)\left\{1+O\left(\frac{1}{u}\right)\right\}, \tag{5.7}$$

and
$$\frac{1}{\mathrm{Re}\,T_2}+\frac{1}{\mathrm{Re}\,T_3} = (2u)^{\frac{1}{4}}m^{-\frac{1}{2}}k(u-m)\,O\left(\frac{1}{u}\right). \tag{5.8}$$

From (5.2), (5.8) and the fact that

$$k(u-m)\,\mathsf{E}^{-1}(u-m, \zeta\sqrt{2u}) \leqslant \mathsf{E}(u-m, \zeta\sqrt{2u}),$$

it is perceivable that the whole contribution from the first solution on the left-hand side of (5.3) is absorbable in the O-estimate for the error term on the right-hand side. Then on replacement of x by $-x$ and use of (5.7) we are led to (5.2) with ϵ_1 given by (5.6), as asserted.

Another useful check is to compare the limiting form of (5.2) and (5.6) as $\zeta \to -\infty$ with the known behaviour of the Ferrers functions as $x \to -1$. The writer has confirmed that the agreement is satisfactory, but again details are too lengthy to warrant reproduction.

6. Imaginary parameters and imaginary turning points

This case is related to that of § 4 in the same way that § 5 is related to §§ 2 and 3. After applying the Liouville transformation we obtain the differential equation

$$d^2w/d\zeta^2 = \{-u^2(\zeta^2+\alpha^2)+\psi(\alpha,\zeta)\}\,w, \tag{6.1}$$

with solutions

$$w = (x^2+a^2)^{\frac{1}{4}}\,(\zeta^2+\alpha^2)^{-\frac{1}{4}}\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(\pm x).$$

Here $$0 < u \leqslant m, \quad a^2 = \frac{m^2}{u^2}-1, \quad \tfrac{1}{2}\alpha^2 = (1+a^2)^{\frac{1}{2}}-1 = \frac{m}{u}-1,$$

and ζ and $\psi(\alpha,\zeta)$ are given by (4.4) and (4.6), respectively.

Equation (6.1) is in the standard form of theorem IV of Olver (1975, § 10.1). Identifying the solutions by letting $x \to 1-$ in the manner of previous sections, we arrive at

$$\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(x)\} = \frac{m^{\frac{1}{2}}}{(2u)^{\frac{1}{4}}}\left(\frac{\zeta^2+\alpha^2}{x^2+a^2}\right)^{\frac{1}{4}} k^{-\frac{1}{2}}(u-m)\,\{W(u-m,\zeta\sqrt{2u})+\epsilon\}, \tag{6.2}$$

where T_1 is defined in § 5 above. The error term ϵ is bounded as in (10.3) of Olver (1975), this bound being valid when $0 < u \leqslant m$ and $-1 < x < 1$. Equation (6.2) should be compared with (5.2).

For large u, $$\epsilon = \mathsf{M}(u-m,\zeta\sqrt{2u})\,O(u^{-1}\ln u), \tag{6.3}$$

uniformly for $m \in [u, u+\Delta u]$ and $x \in (-1,1)$, Δ again denoting any fixed positive number.

Corresponding results for other Ferrers functions can be constructed by straightforward use of connection formulae, since $\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(x)\}$ and $\mathrm{Re}\,\{T_1\,\mathrm{P}^{\mathrm{i}m}_{-\frac{1}{2}+\mathrm{i}u}(-x)\}$ comprise a numerically satisfactory pair of solutions in the present circumstances.

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